Error propagation in extended chaotic systems

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Abstract. A strong analogy is found between the evolution of localized disturbances in extended chaotic systems and the propagation of fronts separating different phases. A condition for the evolution to be controlled by nonlinear mechanisms is derived on the basis of this relationship. An approximate expression for the nonlinear velocity is also determined by extending the concept of the Lyapunov exponent to a growth rate of finite perturbations.

In recent years the study of front propagation in spatially extended systems has known a renewed interest, due to the relevance of spreading fronts for the emergence of spatial structures (patterns) in non-equilibrium systems [1]. In particular, simple reaction–diffusion models seem to be appropriate for describing propagation phenomena in different fields, such as fluid dynamics, liquid crystals [2], epidemics [3], chemical reactions, crystal growth [4] and biological aggregation [5]. Several mathematical models which describe the spreading of a disturbance into unstable (or metastable) steady states have been studied in detail, in order to uncover the mechanisms underlying the propagation of fronts [6–9].

The main result of these studies can be summarized with reference to the one-dimensional equation

$$u_t = u_{xx} + g(u) \tag{1}$$

where $g(u) \in C^1[0, 1]$, $g(0) = g(1) = 0$. If $g > 0$ in $(0, 1)$, then $u = 0$ is an unstable fixed point, while $u = 1$ is a stable one. In this case, any sufficiently localized initial perturbation $u(x, t = 0)$ generates a propagating front joining the unstable to the stable state (figure 1). A linear stability analysis shows that the front can have any speed v_F larger than a minimal value v_L which depends on the behaviour of $g(u)$ at $u = 0$. However, very often velocities larger than v_L require special initial conditions to be realized, so that the 'physical' speed is exactly $v_F = v_L$. In the following we shall call v_L the *linear velocity*. Whether it is selected or not depends on the behaviour of $g(u)$ for $u > 0$. In particular, it has been shown in [10] that convexity of $g(u)$ is sufficient for $v_F = v_L$. Intuitively, we can say that, if $g'(0) > g'(u)$ for all $u > 0$, the front is 'pulled' by the initial growth of u and, otherwise, it is 'pushed' by the faster growth of finite u [11].

In the present paper we consider a different problem, namely the propagation of some perturbation in a chaotic system (figure 2). Thus the front does not separate two different phases, since the system is chaotic (and hence unstable) on both sides of the front. More precisely, we consider two realizations of a one-dimensional coupled map lattice (CML) [12] which differ only locally in the initial conditions, and we watch the spreading of the relative deviation. In spite of the obvious difference with the situation discussed above, we will show that there are surprising similarities. In particular, the derivative $g'(u = 0)$ will be replaced
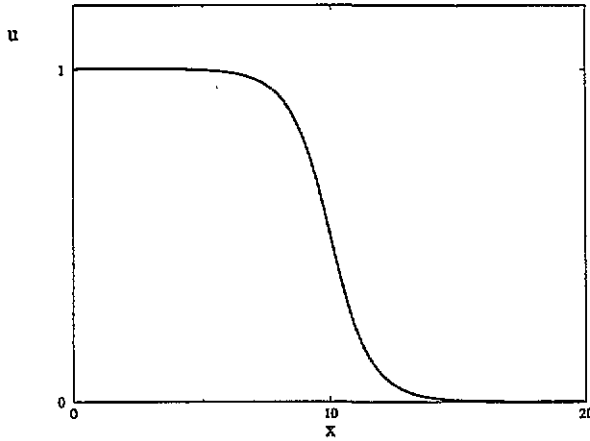
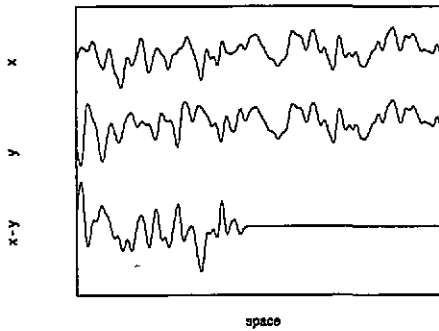**Figure 1.** A typical front connecting an unstable with a stable region.

Figure 2. A typical chaotic state (x), a perturbed state (y), and their difference. The front separates the perturbed from the not yet perturbed region.

by the Lyapunov exponent. In order to formulate (heuristically) a condition equivalent to $g'(0) > g'(u)$, we will introduce a new indicator of the sensitivity to finite perturbations. We shall see that there exists again a minimal velocity v_L, and that the 'physical' velocity v_F can be larger that v_L only if this indicator grows with the perturbation.

The CML is written as

$$x_i^{n+1} = f(\tilde{x}_i^n) \tag{2}$$
$$\tilde{x}_i^n = (1 - \varepsilon)x_i^n + \tfrac{1}{2}\varepsilon(x_{i-1}^n + x_{i+1}^n) \tag{3}$$

where i and n indicate the discrete space and time variables, and ε the diffusive coupling parameter. We use periodic boundary conditions on a chain of length L, $x_i^n = x_{i\pm L}^n$. The function $f(x)$ is assumed to be a map of some interval into itself. We have chosen this for numerical convenience. We are confident that the basic features can be generalized to continuous systems. Disturbances spreading with $v_F \geqslant v_L$ in CMLs have been observed for the first time in [13]. Here, instead of considering the map $G(x)$ studied in [13], we shall discuss two simpler examples: the 'generalized Bernoulli shift'

$$f(x) = rx \qquad \bmod 1 \tag{4}$$

and the circle map

$$f(x) = (x + \alpha) \qquad \text{mod } 1. \tag{5}$$

The propagation of infinitesimal disturbances is governed by the evolution in tangent space:

$$u_i^{n+1} = f'(\tilde{x}_i^n)[(1 - \varepsilon)u_i^n + \tfrac{1}{2}\varepsilon(u_{i+1}^n + u_{i-1}^n)] \tag{6}$$

where $f' = \mathrm{d}f/\mathrm{d}x$. Instead of considering an initially localized perturbation, we shall first refer to a perturbation decaying exponentially for $i \to \infty$,

$$u_i^0 \sim e^{-\mu i}. \tag{7}$$

Its temporal growth depends on μ,

$$u_i^n \sim e^{\lambda(\mu)n - \mu i}. \tag{8}$$

The position of the front is defined as the rightmost site where u_i^n is larger than some arbitrarily fixed constant $\mathcal{O}(1)$. This gives for its velocity

$$V(\mu) = \frac{\mathrm{d}i}{\mathrm{d}n} = \frac{\lambda(\mu)}{\mu}. \tag{9}$$

For an absolutely unstable system we have $\lambda(\mu = 0) > 0$†, so that $V(\mu)$ diverges for $\mu \to 0$. This is intuitively obvious: an almost flat front will appear to move with arbitrarily large velocity. On the other hand, it can be shown [14] that, for nearest-neighbour coupling, $V(\mu) \to 1$ for $\mu \to \infty$.

We now want to determine the speed v_L when the initial perturbation is localized near $i = 0$ and still infinitesimal (the case of finite perturbations will be discussed later). Since we expect that any front will have a leading edge where it is infinitesimal and exponentially decaying with some exponent μ_0, we have $v_L = V(\mu_0)$.

To determine μ_0 and v_L, we need the (maximal) *co-moving* Lyapunov exponent $\Lambda(v)$ [15]. For a given v, this gives the local growth rate of a disturbance in a reference frame moving with velocity v, $u_i^n \sim e^{\Lambda(v)n}$ if $i = vn$. The selected front speed is such that a disturbance neither grows nor decreases at v_L, i.e. $\Lambda(v_L) = 0$. In order to express this in terms of $\lambda(\mu)$ and μ, we recall that they are related to $\Lambda(v)$ through the Legendre transformation [16, 17]

$$\Lambda(v) = \lambda(\mu) - \mu\frac{\mathrm{d}\lambda(\mu)}{\mathrm{d}\mu} \qquad v = \frac{\mathrm{d}\lambda(\mu)}{\mathrm{d}\mu}. \tag{10}$$

Therefore, the derivative of $V(\mu)$ is directly related to the co-moving exponent;

$$\frac{\mathrm{d}V}{\mathrm{d}\mu} = \frac{1}{\mu}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} - \frac{\lambda}{\mu}\right) = -\frac{\Lambda(v)}{\mu^2}. \tag{11}$$

Using $\Lambda(v_L) = 0$, we now see that $\mathrm{d}V/\mathrm{d}\mu = 0$ at a value μ_0 for which $v(\mu) = v_L$, and since $\Lambda(v)$ is convex (being a Legendre transform), this will be the unique minimum of $V(\mu)$. Finally, we can write

$$v_L = \frac{\lambda(\mu_0)}{\mu_0} = \left(\frac{\mathrm{d}\lambda(\mu)}{\mathrm{d}\mu}\right)_{\mu=\mu_0}. \tag{12}$$

Thus as long as we can consider a perturbation as infinitesimal, it is the lowest possible speed which is selected, which justifies us calling it the 'linear velocity'.

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† Throughout this paper, all Lyapunov exponents are maximal ones, and all perturbations are assumed to be typical so that they grow with maximal rate. There exist of course also atypical perturbations the growth of which is governed by non-leading Lyapunov exponents [14], but they will be neglected.

This expression for v_L is identical to that found in [8] for the propagation into unstable steady states, provided that $\lambda(\mu)$ and μ are identified with the complex part of the frequency and of the wavevector, respectively. Thus, the relation $\lambda = \lambda(\mu)$ plays essentially the role of a dispersion relation [14, 17].

Recalling that for closed systems, $\Lambda(v)$ is always a decreasing function (limiting us to $v \geqslant 0$ for symmetry reasons) and that $\Lambda(v = 0) = \lambda(0)$ [15], we can readily deduce from equation (12) that v_L is defined if and only if the system is absolutely unstable, i.e. $\lambda(0) > 0$. As can be seen from figure 3, $V(\mu)$ steadily increases with μ and $V(\mu \to 0) \to -\infty$ if the local dynamics is not chaotic ($\lambda(0) < 0$). A negative velocity indicates that the perturbation regresses instead of propagating: the system is absolutely stable.
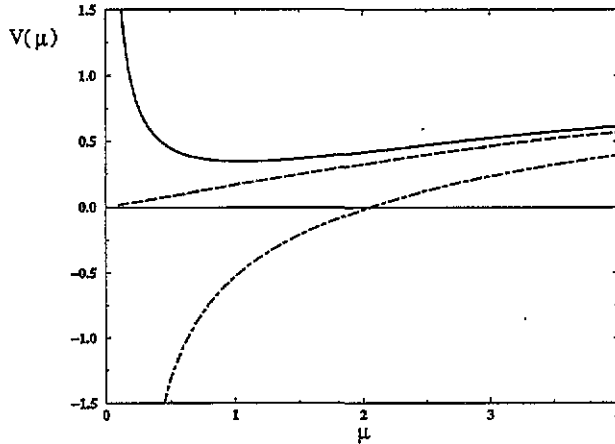

Figure 3. Velocities $V(\mu)$ against μ for the coupled piecewise linear maps (4) with $\varepsilon = 1/3$. The full curve refers to the absolutely unstable situation ($r > 1$), the broken curve to the marginally stable case ($r = 1$) and the dash-dotted one to the absolutely stable case ($r < 1$).

Finally, we consider localized and *finite* initial perturbations. We call the corresponding front velocity v_F. Since any front will have an infinitesimal leading edge, we have to expect that $v_F = V(\mu^*)$ for some value μ^*. It is hard to see how μ^* could be smaller than μ_0, whence we just have to distinguish two possibilities: the 'linear' (or 'pulled') case with $\mu^* = \mu^0$ and $v_F = v_L$, and the 'nonlinear' (or 'pushed') case with $\mu^* > \mu^0$, $v_F > v_L$.

In order to see which case is realized in a particular model, we simulate two chaotic configurations $\{x_i^n\}$ and $\{y_i^n\}$ initially differing in a limited region of the chain (typically 50 sites in chains of $\geqslant 1024$ sites) and coinciding elsewhere. The front position after n iterations is defined as

$$R(n) = \max\{i : |x_i^n - y_i^n| \geqslant \theta\} \tag{13}$$

where θ is a preassigned threshold $\ll 1$. The front velocity is then defined as

$$v_F = \lim_{n \to \infty} \frac{R(n)}{n}. \tag{14}$$

We have verified that v_F is independent of the amplitude of the initial perturbation δ_0 and of the value of the threshold θ when they are varied from 10^{-14} to 10^{-1}.

In this way we measured v_F and v_L for several CML models and couplings. As expected, we found always $v_F \geqslant v_L$. In most cases, $v_F = v_L$ (this was found for logistic, cubic and tent coupled maps for all tested values of the parameters and of ε), but we have also identified

a class of maps (namely, models (4), (5) and the map G studied in [13]) where the strict inequality $v_F > v_L$ is found to hold. The common characteristic of these maps is that $f'(x)$ exhibits a narrow peak (or even a δ-singularity). Moreover, in system (4) [18] and in [13] a transition between the two above regimes is found upon varying a parameter of the map. For map (5), such a transition cannot occur since v_L is always zero, the map being marginally stable. However, also in this case we can observe a finite v_F for a range of α and ε values. This fact stresses even more that this propagation mechanism is not related to local chaoticity, i.e. to sensitive dependence on local and infinitesimal perturbations. The unpredictability resulting from the spreading of perturbations does not result here from local production of entropy but from entropy transport.

The shape of the moving front has been determined with an accurate average over successive realizations of the front itself, following its evolution along the chain. In the leading edge its asymptotic shape is well approximated by an exponential, with a decay rate μ^* (as expected $v_F = V(\mu^*)$ both in the pulled and pushed case), while the nonlinear part of the front, characterized by an amplitude of order $\mathcal{O}(1)$, reveals a quite oscillating behaviour.

In order to determine when the nonlinear mechanism is likely to prevail against the linear one, we reconsider a heuristic conjecture of van Saarlos [8] for fronts propagating into unstable steady and homogeneous states. He observed that $v_F > v_L$ only if the local growth rate of small but finite perturbations increases with their amplitude.

In our case the linear local growth rate of perturbations is represented in the limit of small coupling ε by the Lyapunov exponent of the single map λ_0, which can be defined as

$$\lambda_0 = \lim_{\delta \to 0} \left\langle \log \left| \frac{f(x + \delta/2) - f(x - \delta/2)}{\delta} \right| \right\rangle = \langle \log |f'(x)| \rangle \tag{15}$$

where $\langle \ldots \rangle$ is the average over the invariant measure of the map. If we are interested in the evolution of finite disturbances Δ the average growth rate will be given by

$$I(\Delta) = \left\langle \log \left| \frac{f(x + \Delta/2) - f(x - \Delta/2)}{\Delta} \right| \right\rangle = \langle A(x, \Delta) \rangle. \tag{16}$$

Obviously, $\lim_{\Delta \to 0} I(\Delta) = \lambda_0$. Let us first consider map (4). There, we have

$$A(x, \Delta) = \begin{cases} \log \left[\dfrac{(1 - r\Delta)}{\Delta} \right] & \text{if } x \in [1/r - \Delta/2, 1/r + \Delta/2] \equiv C(\Delta, r) \\ \lambda_0 = \log(r) & \text{otherwise.} \end{cases}$$

Therefore, the indicator I is given by

$$I(\Delta) = \int_{x \in C} dx \, \nu(x) \log \left[\frac{(1 - r\Delta)}{\Delta} \right] + \int_{x \notin C} dx \, \nu(x) \log(r) \tag{17}$$

where $\nu(x)$ is the invariant measure.

The expression is more compact for the circle map, because there $\lambda_0 = 0$ and the invariant measure is flat, so that

$$I(\Delta) = \Delta \log \left[\frac{(1 - \Delta)}{\Delta} \right]. \tag{18}$$

This is positive for $0 < \Delta < 1/2$, and is an increasing function at small Δ. Therefore at small but finite Δ we have a positive growth rate in spite of the stability against infinitesimal perturbations. This is always the case when we consider maps like (4), (5) or G. Conversely, for all the other maps we looked at (i.e. logistic and tent maps), we found $I(\Delta) < \lambda_0$ for

all finite Δ-values (see figure 4). Accordingly, we can conjecture that whenever a nonlinear propagation mechanism has been observed, the quantity $I(\Delta)$ is an increasing function at small Δ. If, instead, $I(\Delta) < \lambda_0$ for any Δ, propagation in the correspondig CML will take place with velocity $v_F = v_L$ for any coupling constant ε. Nonlinear propagation of perturbations can arise only if finite disturbances are, on average, amplified faster than infinitesimal ones, i.e. by a factor $> \exp[\lambda_0]$ during a single iteration.
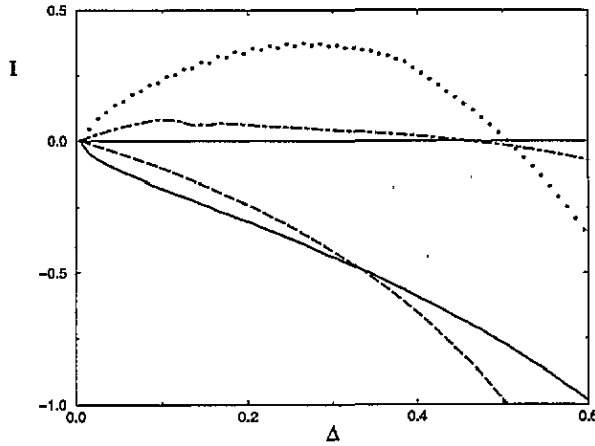

Figure 4. Nonlinearity indicator $I(\Delta)$ for the single maps: logistic map at the crisis (full curve); tent map (broken curve); circle map (5) with $\alpha = [1 - (\sqrt{5} - 1)/2]$ (dotted curve); generalized Bernoulli shift (4) with $r = 1.10$ (dash-dotted curve).

In order to give a quantitative estimate of v_r we have to take into account the coupling between different sites. We have seen that the linear velocity is the minimum value of $V(\mu)$ which is obtained from the growth rate $\lambda(\mu)$. If $v_F > v_L$, the exponential slope μ^* of the leading edge is larger than the value μ_0 where $V(\mu)$ is minimal. The main effect of nonlinearities is to change $\lambda(\mu)$ into a function $\lambda(\mu, \Delta)$ which coincides with it along the leading edge of the front (where Δ is infinitesimal) but becomes different as Δ becomes large. Our main assumption now is that we have just to replace $\lambda(\mu)$ with a suitable average over $\lambda(\mu, \Delta)$. The average has to be taken over the Δ range where $I(\Delta) > \lambda(0)$ and which thus 'pushes' the front.

The main problem in this assumption is that Δ is a fluctuating quantity. In order to make it practically applicable, we have to resort to a mean-field approximation.

By assuming that the perturbation decays exponentially as

$$\Delta_i^n = e^{-\mu i} \Phi_i^n \tag{19}$$

from equations (2) and (3), we find that it evolves in time according to

$$\Delta_i^{n+1} = |f(\tilde{x}_i^n + \tfrac{1}{2}\tilde{\Delta}_i^n) - f(\tilde{x}_i^n - \tfrac{1}{2}\tilde{\Delta}_i^n)| = \tilde{\Delta}_i^n \, e^{A(\tilde{x}_i^n, \tilde{\Delta}_i^n)} \tag{20}$$

where

$$\tilde{\Delta}_i^n = e^{-\mu i}((1 - \varepsilon)\Phi_i^n + \tfrac{1}{2}\varepsilon(\Phi_{i-1}^n \, e^{\mu} + \Phi_{i+1}^n \, e^{-\mu})). \tag{21}$$

We now introduce a mean-field approximation by assuming that Φ_i^n is independent of i, and $A(x, \Delta)$ equal to its average over x. This allows us to rewrite equation (20) as

$$\Phi^{n+1} = \Phi^n[(1 - \varepsilon) + \varepsilon \cosh(\mu)]e^{I(\Delta)}. \tag{22}$$

Performing an average over the range D of Δ where $I(\Delta) > \lambda(0)$, we obtain an effective Lyapunov exponent

$$\lambda_c(\mu) = \log[(1 - \varepsilon) + \varepsilon \cosh(\mu)] + \frac{1}{|D|} \int_D \mathrm{d}\Delta\, I(\Delta) \qquad |D| = \int_D \mathrm{d}\Delta \tag{23}$$

and, in analogy with the linear case,

$$V_c(\mu) = \frac{\lambda_c(\mu)}{\mu}. \tag{24}$$

Just like $V(\mu)$, $V_c(\mu)$ is a convex function with a unique minimum. It is thus natural to assume that the selected velocity for the front will be given by the minimum of (24):

$$v_F = \min_\mu V_c(\mu). \tag{25}$$

The value μ_c where $V_c(\mu)$ is minimal would be equal to μ^* if equation (19) would hold with the same μ in the leading edge and in the pushing region. This, however, need not be the case and we indeed found $\mu_c < \mu^*$ in general.
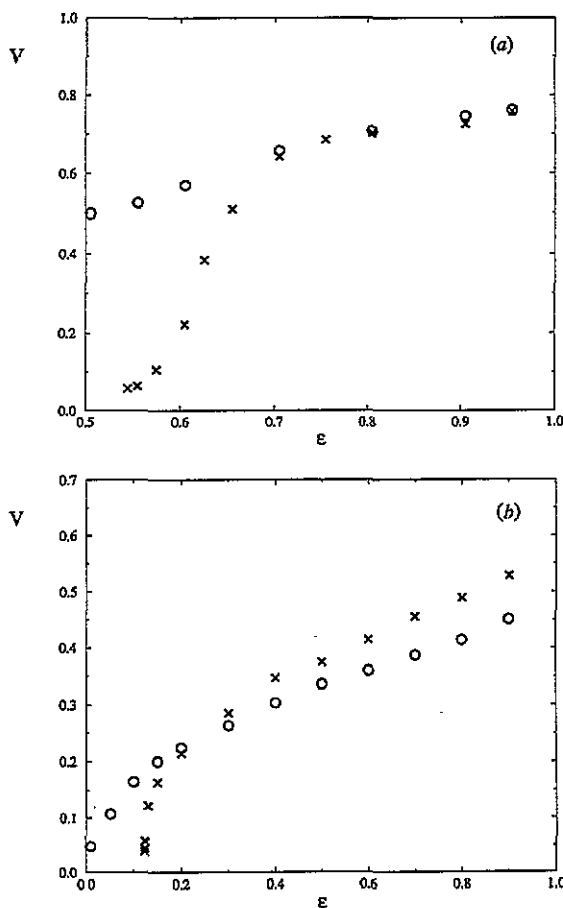

Figure 5. Front velocities for circle coupled maps as a function of the coupling parameter ε: measured velocity v_F (crosses) and theoretical prediction v_T (circles). (a) refers to a map parameter $\alpha = [1 - (\sqrt{5} - 1)/2]$ and (b) to $\alpha = [1 - (\sqrt{5} - 1)/2]/8$.

In figure 5, the numerical results are reported together with the predictions obtained from equation (25) for the circle map with two different values of α. The agreement between simulation and theoretical results is reasonably good for large coupling ε. However, it can be seen that the front propagates only for ε larger than a certain threshold $\varepsilon_c(\alpha)$. Equation (25) does not predict such a transition which can be attributed to the particular structure of the invariant measure for model (5) for $\varepsilon < \varepsilon_c(\alpha)$. The invariant measure becomes extremely irregular below threshold and this no longer allows a 'synchronization' of the motion of the disturbances, as necessary to observe a front propagation. Obviously, this cannot be recovered from a mean-field analysis. An analogous comparison for map (4) with $\varepsilon = 1/3$ is reported in figure 6. The overall behaviour of the velocity provided by equation (25) is in agreement with that of the measured v_F. More precisely, the theoretical predictions are larger than v_L for any value of the parameter r.
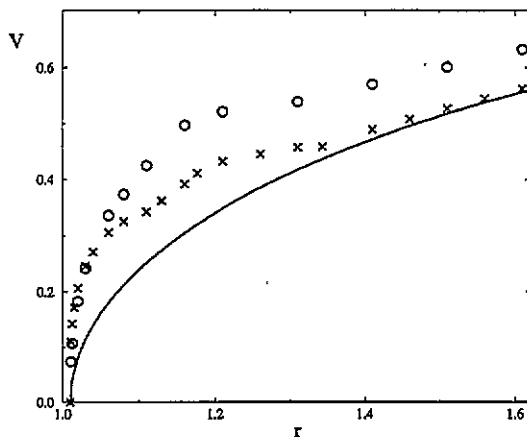

Figure 6. As in figure 5 for a coupled piecewise linear map with $r > 1$ ($\varepsilon = 1/3$). In this case the linear velocity v_L is also reported (full curve), since it is positive.

In conclusion, we have demonstrated that the propagation of perturbations in chaotic systems is very similar to the propagation of fronts between steady states. This includes the possibility of 'nonlinear' selection of velocity. We have verified that an extremely crude estimate of the influence of nonlinearities on the velocity gives surprisingly good agreement with simulations of several coupled map lattices.

As a final remark, we want to point out that the velocity of the information transport is a fundamental ingredient for the characterization of the structure of the invariant measure in spatially extended systems [19]. However, the scaling behaviour of the fractal dimension is usually interpreted within a linear framework [20]. Clearly, when the nonlinear propagation mechanism prevails over the linear one, this approach is no longer sufficient. Work is in progress in order to reveal the distinct effects, due to these two mechanisms, on the structure of strange attractors associated with extended systems.

Acknowledgments

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